ABSTRACT. We prove a converse of Yano's extrapolation theorem for translation invariant operators.

# A CONVERSE EXTRAPOLATION THEOREM FOR TRANSLATION-INVARIANT OPERATORS

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## 1. INTRODUCTION

Let $X$ be a compact symmetric space with compact symmetry group $G$, and $r > 0$, $1 < p_0 < \infty$ be numbers; all constants may depend on $r$ and $p_0$. If a linear operator $T$ is bounded on $L^p$, $1 < p < p_0$ with an operator norm of $O((p-1)^{-r})$ as $p \to 1$, then it is a classical extrapolation theorem of Yano [5] that $T$ also maps $L \log^r L(X)$ to $L^1(X)$.

In this paper we show the following converse:

**Theorem 1.1.** *Let $G$, $X$, $p_0$, and $r$ be as above. Suppose $T$ is translation invariant, maps $L \log^r L$ to $L^1$, and is bounded on $L^{p_0}$. Then $T$ is bounded on $L^p$, $1 < p < p_0$ with an operator norm of $O((p-1)^{-r})$.*

This theorem is false without the assumption of translation invariance, since $L^p$ is not an interpolation space between $L \log^r L$ and $L^{p_0}$. For a concrete counterexample, take $E$ and $F$ be subsets of $X$ of measure $2^{-N}$ and $N^{rp_0'}2^{-N}$ respectively, where $N$ is a large number. Then the operator

$$Tf = 2^N N^{-r/(p_0-1)}\langle f, \chi_E\rangle \chi_F$$

maps $L \log^r L$ to $L^1$ and bounded on $L^{p_0}$, but the $L^p$ operator norm for $1 < p < p_0$ grows polynomially in $N$.

The translation invariance hypothesis is exploited via the following heuristic principle: if $f$ is a function on $X$ supported on a set of measure $O(1/N)$, then there exists $N$ translates of $f$ which are essentially disjoint. This idea is used in factorization theory (see e.g. [1]) and also appears in the abstract theory of covering lemmas (e.g. [2], [3]). The point is that the $(L \log^r L, L^1)$ hypothesis yields more information when applied to the sum of the $N$ translates of $f$ than when applied to just $f$ by itself.

The theorem also holds for $p_0 = \infty$, either by a routine modification of the argument, or by assuming an a priori operator bound on $L^2$ (for instance), applying the theorem with $p_0 = 2$, and re-interpolating the result with $L^\infty$ to obtain a better bound on $L^2$. The theorem also holds of course for $r = 0$ by Riesz convexity.

Although our theorem is phrased for compact spaces, it can be extended to non-compact Lie groups if all operator norms are local. In other words, if $T$ is translation invariant, locally bounded on $L^{p_0}$ and locally maps $L \log^r L$ to $L^1$, then $T$ is also locally bounded on $L^p$, $1 < p < p_0$, with an operator norm of $O((p-1)^{-r})$. This can be proven either by direct modification of the argument, or by abstract transplantation considerations.

As is well known, the space $L \log^r L$ is an atomic space generated by the atoms $|E|^{-1} \log(1/|E|)^{-r} \chi_E$, where $E$ is an arbitrary measure subset of $X$ with $0 < |E| \ll 1$. (For completeness, we provide a proof of this fact in an appendix). As a consequence we have

**Corollary 1.2.** *Let $G$, $X$, $p_0$, $r$ be as above, and let $T$ be a translation invariant operator which is bounded on $L^{p_0}(X)$. Then a necessary and sufficient condition for $T$ to be bounded on $L^p$, $1 < p < p_0$, with an operator norm of $O(1/(p-1)^r)$, is that*

$$\int |T\chi_E| \lesssim |E| \log(1/|E|)^r$$

*for all measurable subsets $E$ of $X$ with $0 < |E| \ll 1$.*

In a subsequent paper with Jim Wright [4], we show that certain classes of rough multipliers are bounded from $L \log^r L$ to $L^1$ for various values of $r$, and apply Theorem 1.1 to deduce sharp bounds for the growth of $L^p$ operator norms.

## 2. The main lemma

We use $A \lesssim B$ to denote the estimate $A \leq CB$ where $C$ is a constant depending on $p_0$, $r$, and the implicit constants in Theorem 1.1, and $A \sim B$ to denote the estimates $B \lesssim A \lesssim B$.

Fix $p$; by Riesz convexity we may assume that $p < \frac{1+p_0}{2}$. All of our implicit constants shall be independent of $p$.

The main lemma in the argument is

**Lemma 2.1.** *Let $E, F$ be subsets of $X$ with $0 < |E| \leq |F|$. Then we have*

$$(1) \qquad \int_F |Tf| \lesssim |E|^{1/p'} \left(\frac{1}{p-1} + \log(2 + \frac{|F|}{|E|})\right)^r \|f\|_p$$

*for all $L^p$ functions $f$ supported on $E$.*

We remark that without translation invariance, one can only obtain (1) with $\log(2 + \frac{|F|}{|E|})$ replaced by $\log(2 + \frac{1}{|E|})$.

**Proof** Fix $E$, $F$, $f$; we may normalize $\|f\|_p = 1$. Let $h$ denote the function $h = |\chi_F Tf|$, and define the quantity $A$ by

$$(2) \qquad \qquad \qquad \|h\|_1 = A|E|^{1/p'};$$

our task is then to show that

$$(3) \qquad \qquad A \lesssim \left(\frac{1}{p-1} + \log(2 + \frac{|F|}{|E|})\right)^r.$$

Let $N$ be the nearest integer to $\varepsilon/|F|$, where $0 < \varepsilon \ll 1$ is a small constant to be chosen later. The first step in the argument is to construct group elements $\Omega_0, \dots, \Omega_N \in G$ such that

$$(4) \qquad \qquad \left\langle \chi_{\bigcup_{j<J} \Omega_j(F)}, h \circ \Omega_J \right\rangle \leq \frac{1}{2} A|E|^{1/p'}$$

and

$$(5) \qquad \langle (\sum_{j<J} |f| \circ \Omega_j)^{p-1}, |f| \circ \Omega_J \rangle \leq 1$$

for all $0 \leq J \leq N$.

Intuitively, (4) asserts that the $h \circ \Omega_j$ are essentially disjoint, while (5) asserts that the $|f| \circ \Omega_j$ are similarly disjoint. For future reference, we note that (5) and the $L^p$ normalization of $f$ implies that

$$(6) \qquad \int_X (\sum_{j \leq J} |f| \circ \Omega_j)^p - \int_X (\sum_{j<J} |f| \circ \Omega_j)^p \leq C.$$

We now construct the desired group elements. We may let $\Omega_0$ be arbitrary since (4), (5) are vacuously true for $J = 0$. Now suppose inductively that $\Omega_0, \ldots \Omega_{J-1}$ have already been constructed for some $0 < J \leq N$ such that (5) (and hence (6)) holds for all previous values of $J$. We will show that

$$(7) \qquad \int_G \langle \chi_{\bigcup_{j<J} \Omega_j(F)}, h \circ \Omega_J \rangle \, d\Omega_J \leq \frac{1}{8} A|E|^{1/p}.$$

and

$$(8) \qquad \int_G \langle (\sum_{j<J} |f| \circ \Omega_j)^{p-1}, |f| \circ \Omega_J \rangle \, d\Omega_J \leq \frac{1}{4}$$

where $d\Omega_J$ is Haar measure on $G$. By Markov's inequality, this implies that a randomly selected $\Omega_J$ has probability at least $3/4$ of obeying (4) and probability at least $3/4$ of obeying (5), and so there exists an $\Omega_J$ with the desired properties.

From Fubini's theorem, (2), and the identity

$$\int_G g \circ \Omega(x) \, d\Omega = C \int_X g$$

for all $x \in X$, the left-hand side of (7) evaluates to

$$C| \bigcup_{j<J} \Omega_j(F)| A|E|^{1/p} \lesssim J|F|A|E|^{1/p} \lesssim \varepsilon A|E|^{1/p}.$$

Thus (7) holds if $\varepsilon$ is sufficiently small. The left-hand side of (8) can similarly be evaluated as

$$C(\int_X (\sum_{j<J} |f| \circ \Omega_j)^{p-1})(\int_X |f|).$$

From Hölder we have

$$\int_X |f| \leq |E|^{1/p'} \|f\|_p = |E|^{1/p'},$$

and

$$\int_X (\sum_{j<J} |f| \circ \Omega_j)^{p-1} \leq (J|E|)^{1/p}(\int_X (\sum_{j<J} |f| \circ \Omega_j)^p)^{\frac{p-1}{p}}.$$

On the other hand, from (6) and the induction hypothesis we have

$$\int_X (\sum_{j<J} |f| \circ \Omega_j)^p \lesssim J.$$

Combining all these estimates, we see that

$$\text{LHS of (8)} \lesssim J|E| \lesssim \varepsilon |E|/|F| \lesssim \varepsilon$$

Thus we obtain (8) if $\varepsilon$ is sufficiently small.

Fix $\varepsilon$; all constants may now implicitly depend on $\varepsilon$. By telescoping (6) we have

$$(9) \qquad\qquad \int_X (\sum_{j \leq N} |f| \circ \Omega_j)^p \lesssim N \lesssim |F|^{-1}.$$

Let $\epsilon_j = \pm 1$ be an arbitrary assignment of signs. Then the function $\sum_{j \leq N} \epsilon_j f \circ \Omega_j$ has a $L^p$ norm of $O(|F|^{-1/p})$ and is supported on a set of measure $O(N|E|) = O(|E|/|F|)$. We now apply

**Lemma 2.2.** *Let $g$ be a function supported on a set $E \subset E$. Then*

$$\|g\|_{L \log^r L} \lesssim (\frac{1}{p-1} + \log(2 + \frac{1}{|E|}))^r |E|^{1/p'} \|g\|_p.$$

**Proof** We divide into two cases, $|E| \geq 2^{-2r/(p-1)}$ and $|E| \leq 2^{-2r/(p-1)}$. We normalize

$$\|g\|_p = (p-1)^r$$

in the first case and

$$\|g\|_p = (\log \frac{1}{|E|})^{-r} |E|^{-1/p'}$$

in the second; in either case our task reduces to showing that

$$\int_E |g| \log(2 + |g|)^r \lesssim 1.$$

We may restrict ourselves to the set

$$E' = \{x \in E : |g(x)| \geq 2 + |E|^{-1} \log^{-r}(2 + \frac{1}{|E|})\},$$

since the contribution outside of $E'$ is clearly acceptable. In this set $\log(2 + |g|)$ may of course be replaced by $\log |g|$.

The function $\frac{\log^r t}{t^{p-1}}$ is increasing for $1 \leq t < e^{r/(p-1)}$ and decreasing for $t > e^{r/(p-1)}$, with a global maximum of $\frac{(r/e)^r}{(p-1)^r}$. We thus have

$$\frac{\log(|g|)^r}{|g|^{p-1}} \lesssim \frac{1}{(p-1)^r}$$

in the first case and

$$\frac{\log(|g|)^r}{|g|^{p-1}} \lesssim |E|^{p-1} \log^{pr} \frac{1}{|E|}$$

if the second case. In either case the claim follows by multiplying this estimate by $|g|^p$ and integrating, using the $L^p$ normalization of $g$. ∎

From this lemma we obtain

$$\|\sum_{j\leq N}\epsilon_j f\circ\Omega_j\|_{L\log^r L} \lesssim |E|^{1/p'}|F|^{-1}(\frac{1}{p-1}+\log(2+\frac{|F|}{|E|}))^r.$$

Since $T$ is translation invariant and maps $L\log^r L$ to $L^1$, we thus have

$$\|\sum_{j\leq N}\epsilon_j Tf\circ\Omega_j\|_1 \lesssim |E|^{1/p'}|F|^{-1}(\frac{1}{p-1}+\log(2+\frac{|F|}{|E|}))^r.$$

Randomizing the signs $\epsilon_j$ and taking expectations using Khinchin's inequality, we obtain

$$\|(\sum_{j\leq N}|Tf\circ\Omega_j|^2)^{1/2}\|_1 \lesssim |E|^{1/p'}|F|^{-1}(\frac{1}{p-1}+\log(2+\frac{|F|}{|E|}))^r.$$

In particular, we have

$$(10) \qquad \|(\sum_{j\leq N}(h\circ\Omega_j)^2)^{1/2}\|_1 \lesssim |E|^{1/p'}|F|^{-1}(\frac{1}{p-1}+\log(2+\frac{|F|}{|E|}))^r.$$

If we integrate the trivial pointwise estimate

$$(\sum_{j\leq J}(h\circ\Omega_j)^2)^{1/2} \geq (\sum_{j<J}(h\circ\Omega_j)^2)^{1/2} + h\circ\Omega_J(1-\chi_{\bigcup_{j<J}\Omega_j(F)})$$

using (2) and (4), we obtain

$$\|(\sum_{j\leq J}(h\circ\Omega_j)^2)^{1/2}\|_1 \geq \|(\sum_{j<J}(h\circ\Omega_j)^2)^{1/2}\|_1 + \frac{1}{2}A|E|^{1/p'}.$$

Telescoping this for all $1\leq J\leq N$, we obtain

$$\|(\sum_{j\leq N}(h\circ\Omega_j)^2)^{1/2}\|_1 \geq \frac{1}{2}NA|E|^{1/p'} \sim A|E|^{1/p'}|F|^{-1}.$$

Comparing this with (10) we obtain (3) as desired. ∎

## 3. Conclusion of the argument

We are now ready to prove Theorem 1.1. By duality, it suffices to prove the bilinear form estimate

$$(11) \qquad |\langle Tf, g\rangle| \lesssim \frac{1}{(p-1)^r}$$

for all $f, g$ such that $\|f\|_p = 1$, $\|g\|_{p'} = 1$.

Fix $f, g$; we may assume that $f, g$ are non-negative. Let $f^*: \mathbf{R}^+ \to \mathbf{R}^+$ be the non-increasing left-continuous re-arrangement of $f$, so that $\|f^*\|_p = 1$ and

$$(12) \qquad |\{x : f(x) > f^*(\alpha)\}| \leq \alpha.$$

Similarly define $g^*$.

For any integers $q \geq 1$ and $k < C$, define $f_{k,q}$ to be the restriction of $f$ to the set $\{x : f^*(2^{qk+q}) < f(x) \leq f^*(2^{qk})\}$. Since $X$ has finite measure, we thus have

$$f = \sum_k f_{k,q}.$$

Similarly define $g_{k,q}$.

Usually one takes $q = 1$, but because of our desire for sharp bounds as $p \to 1$ it shall be more appropriate to choose $q$ so that $q \sim 1/(p-1) \sim p'$.

By the triangle inequality, (11) will now follow from the estimates

$$\tag{13} \sum_{k,l:k\geq l+1} |\langle Tf_{k,q}, g_{l,q}\rangle| \lesssim 1$$

and

$$\tag{14} \sum_{k,l:k\leq l} |\langle Tf_{k,q}, g_{l,q}\rangle| \lesssim q^r.$$

Let us first prove (13). By splitting

$$f_{k,q} = \sum_{k'=qk}^{qk+q-1} f_{k',1}, \quad g_{l,q} = \sum_{l'=ql}^{ql+q-1} g_{l',1}$$

we see that the left-hand side of (13) is majorized by

$$\sum_{k',l':k'>l'} |\langle Tf_{k',1}, g_{l',1}\rangle|.$$

Since $T$ is bounded on $L^{p_0}$, we may use Hölder's inequality to majorize this by

$$\sum_{k',l':k'>l'} \|f_{k',1}\|_{p_0} \|g_{l',1}\|_{p_0'}.$$

From (12) and the definition of $f_{k,q}$ we have

$$\|f_{k',1}\|_{p_0} \lesssim 2^{k'/p_0} f^*(2^{k'})$$

and similarly

$$\|g_{l',1}\|_{p_0} \lesssim 2^{l'/p_0'} g^*(2^{l'}).$$

Thus the left-hand side of (13) is majorized by

$$\sum_{k',l':k'>l'} (2^{k'/p} f^*(2^{k'}))(2^{l'/p'} g^*(2^{l'})) 2^{-(k'-l')(\frac{1}{p}-\frac{1}{p_0})}.$$

The estimate (13) then follows from Young's inequality for bilinear forms. Indeed, the first expression in parentheses has an $l^p$ norm comparable to $\|f^*\|_p = 1$, the second expression in parentheses has an $l^{p'}$ norm comparable to $\|g^*\|_{p'} = 1$, and convolution kernel is summable with $l^1$ norm of $O(1)$ since we are assuming $p < (1+p_0)/2$.

It remains to prove (14). From Lemma 2.1 and (12) we have

$$\int_{g>g^*(2^{ql+q})} |Tf_{k,q}| \lesssim (2^{qk+q})^{1/p'} (\frac{1}{p-1} + \log(2 + \frac{2^{ql+q}}{2^{qk+q}}))^r \|f_{k,q}\|_p.$$

From the definition of $q$ and the assumptions on $k$, $l$, this simplifies to

$$\int_{g > g^*(2^{ql+q})} |Tf_{k,q}| \lesssim 2^{qk/p'} q^r (1+l-k)^r \|f_{k,q}\|_p.$$

From Hölder's inequality we thus have

$$|\langle Tf_{k,q}, g_{k,q}\rangle| \lesssim g^*(2^{ql}) 2^{qk/p'} q^r (1+l-k)^r \|f_{k,q}\|_p.$$

Thus the left-hand side of (14) is majorized by

$$q^r \sum_{k,l:k \leq l} \|f_{k,q}\|_p (2^{ql/p'} g^*(2^{ql})) 2^{-q(l-k)/p'} (1+l-k)^r.$$

The claim then follows again from Young's inequality and the choice of $q$, since the sequence $\|f_{k,q}\|_p$ is in $l^p$, the sequence $2^{-ql/p'} g^*(2^{ql})$ has an $l^{p'}$ norm comparable to $\|g^*\|_{p'} = 1$, and the convolution kernel is integrable uniformly in $p$. ∎

## 4. APPENDIX: ATOMIC DECOMPOSITION OF ORLICZ SPACES

In this section we show that every $L \log^r L(X)$ function $f$ can be decomposed into a convex linear combination of atoms $|E|^{-1} \log(1/|E|)^r \chi_E$ with $0 < |E| \ll 1$.

We first observe that any function $f$ supported on a set of measure $2^{-k}$ and having a sup norm of $k^{-r} 2^{-k}$ can easily be decomposed in this manner, since bounded functions can be written as convex linear combinations of characteristic functions.

Now let $f$ be a general $L \log^r L(X)$ function; we may normalize so that

$$\int_X |f| \log^r(2 + |f|) = 1.$$

We may also assume without loss of generality that $f$ is non-negative and is supported on a set of measure $\ll 1$.

Let $f^*$ and $f_{k,q}$ be as before. For each integer $k < -C$, we define

$$c_k = |k|^r 2^k f^*(2^k)$$

and

$$a_k(x) = f_{k,1}/c_k.$$

Clearly $f = \sum_{k < -C} c_k a_k$. From (12) and the previous discussion, the $a_k$ are convex linear combinations of atoms uniformly in $k$, so it suffices to show that the $c_k$ are summable, i.e. that

$$\sum_{k < -C} |k|^r 2^k f^*(2^k) \lesssim 1.$$

Since $f$ is non-decreasing, we may bound this expression by

$$C \int_{0 \leq t \ll 1} f^*(t) \log(1/t)^{-r} \, dt.$$

The portion of the integral where $f^*(t) \leq t^{-1/2}$ is clearly acceptable, so we may assume that $f^*(t) \geq t^{-1/2}$. But then we may estimate the above by

$$C \int_{0 \leq t \ll 1} f^*(t) \log(2 + f^*(t))^{-r} \, dt = C \int f \log^r(2 + f) = C$$

as desired.

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